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HARMONIC MEASURES AND BOWEN-MARGULIS MEASURES

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ABSTRACT

We compare two families of measures defined on the absolute of the universal cover of a compact negatively-curved manifold: the harmonic measures and the Bowen-Margulis measures.

Consider a compact negatively-curved Riemannian manifold M. Global geometric properties of M are expressed through families of measures on the absolute of the universal cover of M: two natural families are the family of harmonic measures, which is defined by the potential theory of the Laplacian, and the family of Bowen–Margulis measures, which is defined by the dynamics of the geodesic flow.

These two families are very similar and here we prove that in dimension 2, the corresponding measures either are singular or coincide, and they coincide only when the curvature is constant (Theorem 2). The property that they do coincide has several geometric characterizations (Theorem 1). They can be summarized by saying that asymptotic quantities on the universal cover behave as if they would depend only on the distance. For this reason we call the metric with these properties *asymptotically harmonic.*

A natural question is whether it is always possible to modify a metric with negative curvature into another one which is asymptotically harmonic. The equivalence between harmonic and Bowen–Margulis measures might be easy to realize: one has to realize the maximum value of an upper semi-continuous functional ($\beta/\alpha h$ in the notations of section 5) and this maximum value has to be 1 (see [L]). Therefore the problem of interest is whether, like in dimension 2, this equivalence implies that

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the metric is asymptotically harmonic (see also [K2]). The main aim of this paper is to present basic properties related to this latter problem.

0. Notations

Throughout the paper we consider a compact connected n-dimensional Riemannian manifold M, with negative sectional curvatures. We denote by:

SM the spherical bundle of M,

$p : SM \to M$ the projection, $S_xM = p^{-1}\{x\}$,

for X in SM, γ_X the geodesic on M defined by $(\gamma_X(0), \gamma_X'(0) = X$,

$\{\varphi_t,\ t \in \mathbf{R}\}$ the geodesic flow on SM, defined by $\varphi_t X = (\gamma_X(t), \gamma_X'(t))$, $X \in SM$, $t \in \mathbf{R}$.

$\tilde{M}$ the universal cover of M,

$\pi : \tilde{M} \to M$ the natural projection,

d the distance function on M and on $\tilde{M}$,

$S\tilde{M}$ the spherical bundle of $\tilde{M}$,

for $\tilde{X} \in S\tilde{M}$, $\gamma_{\tilde{X}}$ the geodesic on $\tilde{M}$ such that $\pi_{\gamma_{\tilde{X}}} = \gamma_{\pi\tilde{X}}$ and $\gamma_{\tilde{X}}(0) = p\tilde{X}$,

$\tilde{M}(\infty)$ the absolute of $\tilde{M}$, space of equivalence classes of geodesics for the relation $\sup\{d(\gamma(t), \gamma'(t), t \geq 0\} < +\infty$,

$\tau : S\tilde{M} \to \tilde{M}(\infty)$ the map which associates to a vector $\tilde{X}$ the equivalence class of the geodesic $\gamma_{\tilde{X}}$,

for x in $\tilde{M}$, $\tau_x : S_x\tilde{M} \to \tilde{M}(\infty)$ the restriction of τ to $S_x\tilde{M}$.

The maps τ_x are homeomorphisms between $S_x\tilde{M}$ and $\tilde{M}(\infty)$ and we often write (x, ξ) for the element $\tilde{X}$ in $S\tilde{M}$ such that

$$p\tilde{X} = x, \qquad \tau\tilde{X} = \xi.$$

There is a natural topology on $\tilde{M} \cup \tilde{M}(\infty)$ which makes $\tilde{M}(\infty)$ the boundary of $\tilde{M} \cup \tilde{M}(\infty)$.

Let $\tilde{X} \in S\tilde{M}$; we define the Busemann function $\psi_{\tilde{X}}$ on $\tilde{M}$ by

$$\psi_{\tilde{X}}(y) = \lim_{t \to \infty} d(y, \gamma_{\tilde{X}}(t)) - t.$$

For each fixed $\tilde{X}$, $\psi_{\tilde{X}}$ is a C^2 function on $\tilde{M}$.

Let Δ be the Laplace Beltrami operator on $\tilde{M}$ and define $B(y, \tau\tilde{X}) = \Delta_y \psi_{\tilde{X}}(y)$. The function B has a geometric meaning: $B(y, \xi)$ is the mean curvature of the horosphere associated to ξ passing through y (see e.g. [P]). We denote again by B the factor function on SM.

We shall consider two growth rates on $\tilde{M}$: We denote by h the topological entropy of the geodesic flow and by λ_1 the bottom of the spectrum of the operator $-\Delta$ on $L^2(\tilde{M})$.

Since the functions $e^{-sd(x,y)}$ belong to $L^2(\tilde{M})$ for $s > h/2$ ([M1]), we have:

$$4\lambda_1 \le h^2.$$

Throughout the paper the space SM is endowed with its natural metric and we denote by $\bar{m}$ the normalized Lebesgue measure on SM. The measure $\bar{m}$ is invariant under the geodesic flow.

1. Harmonic measures and Bowen–Margulis measures

The family of harmonic measures solves the Dirichlet problem on $\tilde{M}$. Let f be a continuous function on $\tilde{M}(\infty)$. By [A], [S], there exists a unique function u_f on $\tilde{M} \cup \tilde{M}(\infty)$ such that

$$\Delta u_f = 0 \quad \text{on } \tilde{M} \qquad \text{and}$$

$$u_f(z) \to f(\xi) \qquad \text{when } z \to \xi, \quad \xi \in \tilde{M}(\infty).$$

For all x in $\tilde{M}$, the map $f \to u_f\,(x)$ is a positive linear functional on $C(\tilde{M}(\infty))$. It defines a probability measure on $\tilde{M}(\infty)$, the harmonic measure $\tilde{\mu}_x$. For x, y in $\tilde{M}$, the harmonic measures $\tilde{\mu}_x$ and $\tilde{\mu}_y$ are equivalent, i.e. they have the same negligible sets. The Radon-Nikodym derivative $d\tilde{\mu}_x/d\tilde{\mu}_y$ is given by the following result of [AS].

Let ξ be a point in $\tilde{M}(\infty)$. There exists a unique function $k(x,y,\xi)$ such that

$$k(x,x,\xi) = 1, \qquad \Delta_y k(x,y,\xi) = 0,$$

$$k(x,z,\xi) \to 0 \qquad \text{when } z \to \xi', \quad \xi' \neq \xi \in \tilde{M}(\infty).$$

Then, for all x, y in $\tilde{M}$, $\tilde{\mu}$-a.e. ξ in $\tilde{M}(\infty)$, we have:

$$\frac{d\tilde{\mu}_y}{d\tilde{\mu}_x}(\xi) = k(x,y,\xi).$$

The projection of the measure $\tau_x^{-1}\tilde{\mu}_x$ on $S_{\pi x}M$ does not depend on the choice of the lift of πx. We denote it by $\mu_{\pi x}$.

We define first the family of Bowen–Margulis measures on the spheres S_yM, $y \in M$ and then lift them on $\tilde{M}(\infty)$ (see [H], [K2] for direct definitions).

Let W^{ss} be the strong stable foliation on SM, defined by its equivalence relation:

$$XW^{ss}Y \Leftrightarrow \lim_{t \to +\infty} d(\varphi_t X, \varphi_t Y) = 0.$$

It is a continuous foliation, the leaves of which are C^2-imbedded $(n-1)$-dimensional euclidean spaces. By [BM] there exists a unique (up to scalar multiplication) family of measures on the transversals to the foliation which is invariant under the pseudo group of holonomy maps. For y in M, $\delta > 0$, consider a transversal T as follows:

$$T = \bigcup_{-\delta \le s \le \delta} \varphi_t A,$$

where A is a small open subset of $S_y M$. For δ small enough, the invariant measure ν_T has the following decomposition:

$$(*) \qquad \nu_T = e^{hs}\, ds\, d\nu_A$$

for some measure ν_A on A, independent (up to a scalar constant) of δ (see section 3). Since $S_y M$ is a connected compact set, there exists a unique probability measure ν_y on $S_y M$ such that for all small enough open subsets A of $S_y M$, the restriction of the measure ν_y to A is proportional to the measure ν_A in (*).

Let $x \in \tilde{M}$. We define the Bowen–Margulis measure ν_x on $\tilde{M}(\infty)$ by

$$\nu_x(B) = \nu_{\pi x}(D\pi(\tau_x^{-1}(B))).$$

By definition, the measures $\tilde{\nu}_x$ and $\tilde{\nu}_y$ are equivalent on $\tilde{M}(\infty)$. The Radon–Nikodym derivative is given by:

Proposition 1. *There exists a continuous function F on $\tilde{M}$ such that for all $x, y \in \tilde{M}$, $\tilde{\nu}$-a.e. $\xi \in \tilde{M}(\infty)$.*

$$\frac{d\tilde{\nu}_y}{d\tilde{\nu}_x}(\xi) = e^{-h\psi_{(x,\xi)}(y)} \frac{F(y)}{F(x)}$$

where $\psi_{(x,\xi)}$ is the Busemann function associated to (x, ξ).

See section 3 for a proof.

2. Results

We defined in the preceding section two families of measures on $\tilde{M}(\infty)$ indexed by $\tilde{M}$. In each family the measures are mutually equivalent, and both measure classes are ergodic under the action of $\pi_1(M)$ on $\tilde{M}(\infty)$. There are two possibilities: either for one x—and hence for all x—the measures $\tilde{\mu}_x$ and $\tilde{\nu}_x$ are singular with respect to each other, or for one x—and hence for all x—the measures $\tilde{\mu}_x$ and $\tilde{\nu}_x$ are equivalent. We show below that in dimension 2 in the latter case $\tilde{\mu}_x$ and

$\tilde{\nu}_x$ in fact coincide for all x. Our first result characterizes geometrically this coincidence property.

THEOREM 1. *Let M be a compact connected negatively-curved manifold. With the above notations, the following properties are equivalent*:

(1) *for all x in $\tilde{M}$*, $\tilde{\mu}_x = \tilde{\nu}_x$,

(2) *for all x,y in $\tilde{M}$, ξ in $\tilde{M}(\infty)$*

$$k(x,y,\xi) = \exp(-h\psi_{x,\xi}(y)),$$

(3) *the function B is constant on SM*

and

(4) $4\lambda_1 = h^2$.

When properties (1)-(4) are realized, we say that the metric is *asymptotically harmonic*. For an asymptotically harmonic metric, the measure $\tau_x^{-1}\tilde{\mu}_x$ is the normalized Lebesgue measure on $S_x\tilde{M}$ for all x in $\tilde{M}$, the entropy of the geodesic flow $(SM, \varphi_1;\ \bar{m})$ is h, the constant in property (3) also is h. Clearly when $S\tilde{M}$ is a rank-one semi-simple Lie group the canonical metric is asymptotically harmonic.

THEOREM 2. *Let M be a compact connected negatively curved surface and suppose that for some x in M, the measures μ_x and ν_x are equivalent. Then the curvature is constant.*

It is known that if for some x in M either the measure μ_x or the measure ν_x is equivalent to the Lebesgue measure on S_xM, then the curvature is constant ([Ka1], [Ka2]). Theorem 2 answers the third natural question of this type.

3. Bowen–Margulis measures

In this section we recall some facts behind the statements about Bowen–Margulis measures in section 1. We first have to explain formula (*). In [M2], Margulis constructed a family of measures on the strong unstable manifolds W^{uu} such that a formula analogous to (*) gives a measure on pieces $\tilde{A}^u$ of unstable manifolds,

$$\tilde{A}^u = \bigcup_{-\delta\le s\le\delta} \varphi_s A^{uu}, \qquad A^{uu} \subset W^{uu},$$

which is invariant under the strong stable foliation. By the unique ergodicity of the foliation we can obtain the measure ν_T by sliding along W^{ss} leaves the above measure from pieces of unstable manifolds to subsets of the form

$$\bigcup_{-\delta\le s\le\delta}\varphi_s A,\qquad A\subset S_yM.$$

If δ and A are small enough that this sliding defines a one-to-one map between $\bigcup_{-\delta\le s\le\delta}\varphi_s A$ and a nearby piece of unstable manifold, formula (*) follows.

We now prove Proposition 1. By the construction of the measures $\tilde\nu_x$, we only have to show that for any small open subset A of $\tilde M(\infty)$, for $\tilde\nu$-a.e. ξ,ξ' in A, all x,y in $\tilde M$:

$$(**)\qquad \frac{d\tilde\nu_y}{d\tilde\nu_x}(\xi)\Big/\frac{d\tilde\nu_y}{d\tilde\nu_x}(\xi')=\exp(h\psi_{x,\xi'}(y)-h\psi_{x,\xi}(y)).$$

In fact, if (**) is established and if $W(x,y)$ denotes the normalization factor

$$W(x,y)=\int\exp(-h\psi_{x,\xi}(y))d\tilde\nu_x(\xi),$$

we have for all x,y, $\tilde\nu$-a.e. ξ:

$$\frac{d\tilde\nu_y}{d\tilde\nu_x}(\xi)=\exp(-h\psi_{x,\xi}(y))\,\frac{1}{W(x,y)},$$

the function $(x,y)\to W(x,y)$ is continuous, and $W(x,x)=1$.

There exists a unique (up to scalar multiplication) function F on $\tilde M$ such that $W(x,y)=F(x)/F(y)$. The function F is continuous and Proposition 1 follows.

For a small set A, it is sufficient to prove (**) when x and y are close enough. Suppose now that x and y are close enough that the two subsets of $S\tilde M$, $\bar A=\tau_x^{-1}A$ and $\bar B=\tau_y^{-1}A$, are in the same domain of injectivity of π and, furthermore, that $d(\pi\bar A,\pi\bar B)\le\delta/C$, for some fixed constant C.

The constant C is chosen so that for all Y in $\bar B$, $Y=\pi(y,\xi)$, there is at most one point in $W^{ss}_{\rm loc}(Y)\cap\bigcup_{-\delta\le s\le\delta}\varphi_s\pi\bar A$.

This point is then exactly $\varphi_t\pi(x,\xi)$ where $t=-\psi_{x,\xi}(y)$. In other words, the image under the canonical map associated to W^{ss} of the set $\pi\bar B$ is the graph above a subset of $\pi\bar A$ in $\bigcup_{-\delta\le s\le\delta}\varphi_s\pi\bar A$ of the function: $\pi(x,\xi)\to-\psi_{x,\xi}(y)$. Take $0_\xi, 0_{\xi'}$ neighborhoods of ξ,ξ' in $\tilde M(\infty)$ and consider $\bar 0=\pi\tau_y^{-1}0$. To compute $\nu_{\pi B}(\bar 0)$ we enlarge $\bar 0$ to $\bar{\bar 0}=\bigcup_{-\epsilon\le s\le\epsilon}\varphi_s\bar 0$ and have

$$\nu_T(\bar{\bar 0})=\nu_{\pi B}(\bar 0)\int_{-\epsilon}^{\epsilon}e^{hs}ds.$$

By invariance under W^{ss} we can compute $\nu_T(\bar{\bar{0}})$ (up to a constant) in the set $\bigcup_{-\delta\le s\le\delta} \varphi_s \pi \bar{A}$ and find:

$$\nu_T(\bar{\bar{0}}) = \int_{\{\pi(x,\xi):-\epsilon\le s+\psi_{x,\xi}(y)\le\epsilon\}} e^{hs}\, ds\, d\nu_{\pi\bar{A}}$$

Formula (**) follows from the arbitrariness of ϵ, 0_ξ and $0_{\xi'}$.

4. Equivalent measures

We first have the following proposition:

PROPOSITION 2. *Let M be a compact connected negatively-curved manifold such that for some x in M the measures μ_x and ν_x are equivalent.*

Then there exists a Hölder continuous function U on SM such that for all $\tilde{X}$ in $S\tilde{M}$, we have:

$$\log k(\gamma_{\tilde{X}}(0), \gamma_{\tilde{X}}(1), \gamma_{\tilde{X}}(\infty)) = h + U(\varphi_1 \pi \tilde{X}) - U(\pi \tilde{X}).$$

PROOF. The above LHS is a Γ-invariant Hölder continuous function ([AS] section 6.2).

The quotient function is the opposite of a Hölder continuous function F_0 on SM for which the pressure is 0 and the equilibrium state is the only invariant measure which admits the harmonic measures as a family of transverse measures (see [L]). Under the hypothesis of Proposition 2, this equilibrium state has to be the measure with maximal entropy h. By [Li] (see also [B] Theorem 1.28), there exists a unique Hölder continuous function U on SM such that

$$F_0(X) = -h - U\varphi_1(X) + U(X).$$

COROLLARY 1. *Under the hypothesis of Proposition 2, we have for all x, y in $\tilde{M}$, all ξ in $\tilde{M}(\infty)$.*

$$(***) \qquad \log k(x,y,\xi) = -h\psi_{x,\xi}(y) + U\pi(y,\xi) - U\pi(x,\xi).$$

PROOF. Observe that we have for all t in $\mathbf{R}$, all $\tilde{X}$ in $S\tilde{M}$:

$$\log k(\gamma_{\tilde{X}}(0), \gamma_{\tilde{X}}(t), \gamma_{\tilde{X}}(\infty)) = ht + U(\varphi_t \pi \tilde{X}) - U(\pi \tilde{X}).$$

This formula is true for all $\tilde{X}$ and all t rational number by the uniqueness of the function U given by the proof of Proposition 2. Then for a fixed $\tilde{X}$, both sides are continuous functions of t.

Applying this formula to $((x,\xi),t)$ and $((y,\xi), t-\psi_{x,\xi}(y))$ we get:

$$\log k(x,y,\xi) - \log k(\gamma_{x,\xi}(t),\gamma_{y,\xi}(t-\psi_{x,\xi}(y)),\xi)$$
$$= -h\psi_{x,\xi}(y) + U\pi(y,\xi) - U\pi(x,\xi) + U\pi\gamma_{x,\xi}(t) - U\pi\gamma_{y,\xi}(t-\psi_{x,\xi}(y)).$$

Corollary 1 follows because when t goes to infinity, the distance between $\gamma_{x,\xi}(t)$ and $\gamma_{y,\xi}(t-\psi_{x,\xi}(y))$ goes to zero.

For all fixed ξ, it follows from (∗∗∗) that the function U_ξ defined by

$$U_\xi(y) = U\pi(y,\xi)$$

is a C^2 function on $\tilde{M}$. Differentiating (∗∗∗) yields the following relations.

COROLLARY 2. *Under the hypothesis of Proposition 2, we have for all fixed ξ in $\tilde{M}(\infty)$, all y in $\tilde{M}$,*

$$\operatorname{grad}_y \log k(x,y,\xi) = h\tau_y^{-1}\xi + \operatorname{grad} U_\xi(y),$$
$$-\|\operatorname{grad}_y \log(x,y,\xi)\|^2 = -hB(y,\xi) + \Delta U_\xi(y),$$
$$\Delta U_\xi(y) + h^2 - hB(y,\xi) + \|\operatorname{grad} U_\xi(y)\|^2 + 2h\langle \tau_y^{-1}\xi, \operatorname{grad} U_\xi(y)\rangle = 0.$$

The first line is obtained by taking the gradient of (∗∗∗), the second the Laplacian of (∗∗∗). The third line is obtained by comparing the first two. Theorem 2 follows from Corollary 2 by integrating these relations with respect to the measure $\bar{m}$. We remark that

$$\langle \tau_y^{-1}\xi, \operatorname{grad}_y U_\xi(y)\rangle = \lim_{t\to 0} \frac{1}{t}\left(U_\xi(\gamma_{(y,\xi)}(t)) - U_\xi(y)\right)$$
$$= \lim_{t\to 0} \frac{1}{t}\left(U(\gamma_{y,\xi}(t)) - U(y,\xi)\right)$$
$$= U'(y,\xi).$$

In particular, we have $\int \langle \tau_y^{-1}\xi, \operatorname{grad}_y U_\xi(y)\rangle \, d\bar{m}(y,\xi) = 0$.

We now prove Theorem 2. We can assume that the metric σ on M is given by $\sigma = \rho\sigma_0$ where σ_0 is a metric of constant curvature -1. We identify $(\tilde{M},\sigma_0)$ with the hyperbolic disk and then $k(x,y,\xi)$ is given by the Poisson kernel. Therefore we have

$$\|\operatorname{grad}_y \log k(x,y,\xi)\|^2 = \frac{1}{\rho(\pi y)}.$$

The first relation in Corollary 2 yields:

$$\|\operatorname{grad}_y \log k(x,y,\xi)\|^2 = h^2 + 2hU'(y,\xi) + \|\operatorname{grad} U_\xi(y)\|^2.$$

Integrating with respect to the measure $\bar{m}$ yields

$$\frac{1}{\int \rho dm_0} = h^2 + \int \|\mathrm{grad}_y\, U_\xi(y)\|^2\, d\bar{m}(y,\xi).$$

We used that

$$\int \frac{1}{\rho(y)}\, d\bar{m}(y) = \frac{\int dm_0(y)}{\int \rho(y) dm_0(y)}$$

where m_0 is the normalized Lebesgue measure on (M, σ_0).

We find that $h^2 \leq 1/\int \rho dm_0$. By [Ka1] this is only possible when ρ is constant.

COROLLARY 3. *Under the hypotheses of Proposition* 2, *we have for all x in M, μ_x-a.e. X in s_xM*

$$\frac{d\nu_x}{d\mu_x}(X) = \frac{\exp(-U(X))}{\int_{S_xM} \exp(-U(X))\, d\mu_x(X)}$$

and the function F in Proposition 1 *factorizes to M.*

PROOF. Consider on $\bigcup_{-\delta \leq s \leq \delta} \varphi_s S_x M$ the measure defined by $\exp(hs - U(X)) ds\, d\mu_x(X)$. It follows from (***) that this family of measures is invariant under the W^{ss}-foliation (the details are the same as those of the proof of Proposition 1), and ν_x is given by the above formula. In particular we have

$$\begin{aligned}\frac{d\tilde{\nu}_y}{d\tilde{\nu}_x}(\xi) &= \frac{d\tilde{\mu}_y(\xi)}{d\tilde{\mu}_x(\xi)} \exp(U\pi(x,\xi) - U\pi(y,\xi))\, \frac{F(\pi y)}{F(\pi x)} \\ &= \exp(-h\psi_{x,\xi}(y))\, \frac{F(\pi y)}{F(\pi x)}\end{aligned}$$

where $F(x) = \left(\int \exp(-U(X))\, d\mu_x(X)\right)^{-1}$.

We now begin to prove Theorem 1. Suppose $\mu_x = \nu_x$ for all x. In particular the hypotheses of Proposition 2 are satisfied and, since $d\tilde{\nu}_y/d\tilde{\nu}_x = d\tilde{\mu}_y/d\tilde{\mu}_x$, we have $\log k(x,y,\xi) = -h\psi_{x,\xi}(y) + \log F(\pi y) - \log F(\pi x)$.

In other words, the function U in (***) depends only on pX. The third relation in Corollary 2 then reads: for all X in SM,

$$\Delta U(pX) + h^2 - hB(X) + \|\text{grad } U\|^2 + 2h\, XU = 0.$$

Integrating with respect to the Liouville measure $\bar{m}$, we have

$$h^2 - h\int B(X)d\bar{m}(X) + \int \|\text{grad } U\|^2 dm = 0.$$

By Pesin's formula (see [P]) $\int B d\bar{m}$ is the metric entropy of the Liouville measure and both $h^2 - h\int Bd\bar{m}$ and $\int \|\text{grad } U\|^2 dm$ are non-negative. We must have U constant, that is, $k(x,y,\xi) = \exp(-h\psi_{x,\xi}(y))$. Conversely, when this relation is satisfied, the same proof as for Corollary 3 shows that $\mu_x = \nu_x$ for all x.

The equivalence between the relation $k(x,y,\xi) = \exp(-h\psi_{x,\xi}(y))$ and B constant follows clearly from the expression of the Laplacian in horospherical coordinates: for all $f \in C^2(\mathbf{R},\mathbf{R})$

$$\Delta_y f(\psi_{x,\xi}(y)) = f''(\psi_{x,\xi}(y)) + B(y,\xi)f'(\psi_{x,\xi}(y)).$$

It also follows that the constant is h.

Finally, when B constant equals h, let us prove that $4\lambda_1 = h^2$. We only have to show that $4\lambda_1 \geq h^2$ and this follows, for example, from the fact that the Cheeger isoperimetric constant of $\tilde{M}$ (see [C]) is bigger than h. For we have for all bounded normal domain Ω in $\tilde{M}$, and some x in $\tilde{M}$, ξ in $\tilde{M}(\infty)$:

$$hV(\Omega) = \int_\Omega \Delta\psi_{x,\xi}(y)\, dV(y) = \int_{\partial\Omega} \langle \text{grad } \psi_{x,\xi}(y), \nu\rangle dA \leq A(\partial\Omega).$$

In order to prove the converse we shall use the notion of entropy of the brownian motion on $\tilde{M}$, introduced by V. A. Kaimanovich.

5. Kaimanovich entropy

In [K1] the following properties are proven: let $p(t,x,y)$ be the heat kernel on $\tilde{M}$, the fundamental solution of $\partial u/\partial t = \Delta u$, and m the normalized Lebesgue measure on M.

1. *Let* $\beta = \int(\int \|\text{grad} \log k(x,\cdot,\xi)|\pi^{-1}(y)\|^2 d\tilde{\mu}_{\pi^{-1}y}(\xi))dm(y)$. Then

$$\beta = \lim_{t\to\infty}\left[-\frac{1}{t}\int p(t,x,y)\log p(t,x,y)\, dV(y)\right]$$

where V is the Lebesgue measure on $\tilde{M}$.

2. $\alpha^2 \le \beta \le \alpha h$, where $\alpha = \int B(x,\xi)d\mu_x(\xi)dm(x)$.

PROPOSITION 3. *With the above notations, we have*

$$4\lambda_1 \le \beta.$$

PROOF. We have

$$\begin{aligned}\beta &= \lim_{t\to\infty}\left[-\frac{1}{t}\int p(t,x,y)\log p(t,x,y)\,dV(y)\right]\\ &= \lim_{t\to\infty}\left[-\frac{1}{t}\int_0^t\int\left[(1+\log p(s,x,y)\right]\frac{\partial}{\partial s}p(s,x,y)\,dV(y)ds\right]\\ &= \lim_{t\to\infty}\left[-\frac{1}{t}\int_0^t\int(1+\log p(s,x,y))\Delta_y p(s,x,y)\,dV(y)ds\right]\\ &= \lim_{t\to\infty}\left[\frac{1}{t}\int_0^t\int\frac{\|\nabla p(s,x,y)\|^2}{p(s,x,y)}\,dV(y)ds\right]\\ &= \lim_{t\to\infty}\left[\frac{4}{t}\int_0^t\left(\int\|\nabla\sqrt{p(s,x,y)}\|^2dV(y)\right)ds\right].\end{aligned}$$

Since $\sqrt{p(s,x,y)}$ is a function in $L^2(\tilde{M},V)$ with L^2 norm 1, we have by Rayleigh's principle

$$\beta \ge 4\lim_{t\to\infty}\frac{1}{t}\int_0^t\lambda_1\,ds = 4\lambda_1.$$

We now achieve the proof of Theorem 1 by proving that the relation $4\lambda_1 = h^2$ implies the other properties. In fact we have in this case

$$4\lambda_1 = \alpha^2 = \beta = \alpha h = h^2$$

(compare Proposition 3 and property 2 above).

Since $\beta = \alpha h$, we have equivalence between the measures μ_x and ν_x for all x in M (see [L]). The second and third relations in Corollary 2 are relations between function on $S\tilde{M}$ which are Γ-invariant. We can therefore integrate them on SM with respect to the measure $d\mu_y(\xi)dm(y)$ and get

$$-\beta = -h\alpha + \int\Delta U_\xi(y)\,d\mu_y(\xi)\,dm(y),$$

$$\int\Delta U_\xi(y)\,d\mu_y(\xi)\,dm(y) + h^2 - h\alpha + \int\|\text{grad }U_\xi(y)\|^2d\mu_y(\xi)\,dm(y)$$

$$+\,2h\int\langle\tau_y^{-1}\xi,\text{grad }U_\xi(y)\rangle\,d\mu_y(\xi)\,dm(y) = 0.$$

In the above integrals, $\Delta U_\xi(y)$, $\|\text{grad } U_\xi(y)\|^2$ and $\langle \tau_y^{-1}\xi, \text{grad } U_\xi(y)\rangle$ are the quotient functions on SM. There is a further relation between these integrals, obtained by writing $\int_M \Delta K\, dm = 0$, where K is the quotient function on M of $\int U_\xi(y)\, d\tilde{\mu}_y(\xi)$; we get, using the expression in Corollary 1 for grad $k(x,\cdot,\xi)$,

$$\int (\Delta U_\xi(y) + 2h\langle \tau_y^{-1}\xi, \text{grad } U_\xi(y)\rangle + 2\|\text{grad } U_\xi(y)\|^2)\, d\mu_y(\xi) dm(y) = 0.$$

Using $\beta = h\alpha = h^2$ we get

$$\int \|\text{grad } U_\xi(y)\|^2 d\mu_y(\xi)\, dm(y) = 0,$$

that is, the function U_ξ is constant for μ-a.e. ξ, which means that the function U is constant along a dense set of leaves of the stable foliation. Since U is continuous on SM and the stable foliation is transitive, U is constant. Referring to (***) we have proven that

$$\log k(x,y,\xi) = \exp[-h\psi_{x,\xi}(y)].$$

The above proof provides another criterion for asymptotic harmonicity:

COROLLARY 4. *A compact negatively curved manifold is asymptotically harmonic if and only if*

(5) $\alpha = h$.

So to what extent a metric is asymptotically harmonic can be measured by another continuous dimensionless functional α/h; α is the speed of the brownian motion on $\tilde{M}$, h the volume growth of balls in $\tilde{M}$.

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